NON-ZERO DEGREE MAPS BETWEEN $2N$-MANIFOLDS

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Abstract. Thom-Pontrjagin constructions are used to give a computable necessary and sufficient condition when a homomorphism $\phi : H^n(L;Z) \to H^n(M;Z)$ can be realized by a map $f : M \to L$ of degree k for closed $(n-1)$-connected $2n$-manifolds M and L, $n > 1$. A corollary is that each $(n-1)$-connected $2n$-manifold admits selfmaps of degree larger than 1, $n > 1$.

In the most interesting case of dimension 4, with the additional surgery arguments we give a necessary and sufficient condition for the existence of a degree k map from a closed orientable 4-manifold M to a closed simply connected 4-manifold L in terms of their intersection forms, in particular there is a map $f : M \to L$ of degree 1 if and only if the intersection form of L is isomorphic to a direct summand of that of M.

Keywords: $2n$-manifolds, non-zero degree maps.
Mathematical subject classification 2000: 57R19; 55M25, 57N13, 57R65.

§1. Introduction and Results.

A fundamental question in manifold topology is the following

Question. *For given closed oriented manifolds M and L of the same dimension, can one decide if there is a map $f : M \to L$ of degree $k \neq 0$? in particular $k = 1$?*

For dimension 2 the answer is affirmative and simple (see [1]). But in general this question is difficult. For dimension 3 many rather general facts have been discussed in last ten years (see a survey paper [2]). For dimension at least 4, there are only some

Both author are supported by MSTC, NSFC. The comments of F. Ding, J.Z. Pan, Y. Su and the referee enhance the paper.

special examples discussed up to our knowledge (see references in [3]). In this paper we will obtain affirmative answers for maps between $(n-1)$-connected $2n$-manifolds, $n > 1$, and for maps from arbitrary 4-manifolds to simply connected 4-manifolds.

Unless special indications are made, in this paper all $2n$-manifolds are closed, connected and orientable and $n > 1$. All matrices, homology rings and cohomology rings are in integers.

Let M be a $2n$-manifold and let $\bar{H}^n(M)$ (resp. $\bar{H}_n(M)$) be the free part of $H^n(M)$ (resp. $H_n(M)$). Then the cup product operator

$$\bar{H}^n(M) \otimes \bar{H}^n(M) \to H^{2n}(M) \tag{1.1}$$

defines the intersection form X_M over $\bar{H}^n(M)$, which is bi-linear and unimodular.

Let $\alpha^* = (a_1^*, ..., a_m^*)$ be a basis for $\bar{H}^n(M)$. Then X_M determines an m by m matrix $A = (a_{ij})$, where a_{ij} is given by $a_i^* \cup a_j^* = a_{ij}[M]$, and $[M]$ is the fundamental class of $H^{2n}(M)$. We often present this fact by the formula

$$\alpha^{*\perp} \cup \alpha^* = A[M], \tag{1.2}$$

where $\alpha^{*\perp}$ is the transpose matrix of α^*. We also have

$$\alpha_*{}^{\perp} \cap \alpha_* = A, \tag{1.3}$$

where $\alpha_* \subset \bar{H}_n(M)$ is the Poincaré dual basis of α^*. (We use $\cap$ to denote the intersection of homology classes in this paper.)

Suppose M is a closed $(n-1)$-connected $2n$ manifold. Then M has a presentation $V_{i=1}^m S_i^n \cup_g D^{2n}$, i.e., M is obtained from $V_{i=1}^m S_i^n$, the m-fold wedge of n-spheres with one point y in common, by attaching a $2n$-cell D^{2n} via an attaching map

$$g : S^{2n-1} = \partial D^{2n} \to V_{i=1}^m S_i^n,$$

and the homotopy type of M is determined by $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$. Such a presentation of M provides M a complete set of homotopy invariants $(A; t_1, ..., t_m)$, where $t_i \in \pi_{2n-1}(S^n)$, and $A_{m\times m}$ is the linking matrix. (For details, including the relation between the linking matrix and the intersection matrix, see Section 2).

Suppose L is another closed $(n-1)$-connected $2n$-manifold with presentation $V_{j=1}^l S_j^n \cup_h D^{2n}$ and h corresponding to $(B; u_1, ..., u_l)$.

For each map $f : M \to L$, we use f_* and f^* to denote the induced homomorphisms on the homology rings and cohomology rings. Choose bases $\alpha = \{S_i^n, i = 1, ..., m\}$ and $\beta = \{S_j^n, j = 1, ..., l\}$ of $H_n(M)$ and $H_n(L)$ respectively. Let $P = (p_{ij})$ be an m by l matrix.

Now we can state the following Theorem 1 which gives a necessary and sufficient condition for the existence of a degree k map between $(n-1)$-connected 2n-manifolds M and L, $n > 1$.

Theorem 1. *Suppose M and L are closed oriented $(n-1)$-connected $2n$-manifolds with presentations and data as above. Then there is a map $f : M \to L$ of degree k such that $f_*(\alpha) = \beta P^\perp$ if and only if P satisfies the following equations*

$$P^\perp AP = kB \tag{1.4}$$

and

$$ku_r = \sum_v p_{vr}t_v + (\sum_v \frac{1}{2}p_{vr}(p_{vr}-1)a_{vv} + \sum_{v<w} p_{vr}p_{wr}a_{vw})[s^n, s^n], \tag{1.5}$$

where s^n generates $\pi_n(S^n)$ and $[s^n, s^n] \in \pi_{2n-1}(S^n)$ is the Whitehead product.

Moreover the homotopy classes of maps $f : M \to L$ such that $f_*(\alpha) = \beta P^\perp$ are 1-1 corresponding to the elements of (the finite group) $\pi_{2n}(L)$.

The following Theorem 2 is stated for all maps between two closed oriented $2n$-manifolds, and its proof is direct.

Theorem 2. *Suppose M and L are closed oriented $2n$-manifolds with intersection matrices A and B under some given bases α^* for $\bar{H}^n(M)$ and β^* for $\bar{H}^n(L)$ respectively. If there is a map $f : M \to L$ of degree k such that $f^*(\beta^*) = \alpha^* P$, then*

$$P^\perp AP = kB. \tag{1.6}$$

Moreover if $k = 1$, then X_L is isomorphic to a direct summand of X_M.

In dimension 4, we have the following necessary and sufficient condition for the existence of a degree k map from a closed oriented 4-manifold to a simply connected 4-manifold.

Theorem 3. *Suppose M and L are closed oriented 4-manifolds with intersection matrices A and B under given bases α^* for $\bar{H}^2(M)$ and β^* for $\bar{H}^2(L)$ respectively. If L is simply connected, then there is a map $f : M \to L$ of degree k such that $f^*(\beta^*) = \alpha^* P$ if and only if*

$$P^\perp AP = kB \tag{1.7}$$

Moreover there is a map $f : M \to L$ of degree 1 if and only if X_L is isomorphic to a direct summand of X_M.

Note that $-X_M = X_{-M}$. So similar results can be stated for maps of degree -1.

Corollary 1. *Suppose $f : M \to L$ is a map of degree 1 between two simply connected oriented 4-manifolds. Then M has the homotopy type of $L\#Q$, where Q is a simply connected 4-manifold; Moreover if $\chi(M) = \chi(L)$, then M and L are homotopy equivalent (or homeomorphic if X_M is even).*

Definition: For any two closed oriented n-manifolds M and L, define

$$D(M, L) = \{degf | f : M \to L\}, \tag{1.8}$$

that is, $D(M, L)$ is the set of all degrees of maps from M to L.

Corollary 2. *Suppose M is an $(n-1)$-connected $2n$-manifold, $n > 1$. Let $T(n)$ be the order of the torsion part of $\pi_{2n-1}(S^n)$. Then $k^2 \in D(M, M)$ if k is a multiple of $2T(n)$ when $T(n)$ is even, or a multiple of $T(n)$ otherwise. Moreover if M is a simply connected 4-manifold, then $D(M, M)$ contains all squares of integers.*

Corollary 3. *Suppose M is a closed oriented $2n$-manifold and $dim\bar{H}^n(M) = m$ is odd. Then for any map $f : M \to M$, the degree k of f is a square of an integer.*

Say the intersection form X_M is even if for any $x \in \bar{H}_n(M)$, $x \cap x$ is even.

Corollary 4. *Let $f : M \to L$ be a map of degree k between closed oriented $2n$-manifolds. If X_M is even and X_L is not, then k must be even.*

Definition: Let M, L be two n-dimensional closed oriented manifolds. Say M dominates L if there is a map $f : M \to L$ of non-zero degree.

Corollary 5. *The homotopy types of $(n-1)$-connected $2n$-manifolds dominated by any given closed oriented $2n$-manifold M are finite. Each closed oriented 4-manifold dominates at most finitely many simply connected 4-manifolds.*

The paper is organized as follow: In section 2 we discuss related facts about Thom-Pontrjagin constructions and the presentations of $(n-1)$-connected $2n$-manifolds, which will be used to prove Theorem 1 in Section 3. Theorem 2 and Theorem 3 are proved in Section 4 and Section 5 respectively, the corollaries are proved in Section 6. In Section 7 we give some applications to concrete examples in dimension 4.

We end the introduction by comparing this paper with [3], another paper of the same authors. The major part of Theorem 1 in this paper is the same as Theorem A in [3], which is proved in this paper by using Thom-Pontrjagin construction, a rather elementary and geometric argument, as presented below, and is proved in [3] by more advanced algebraic topology argument. The remaining parts of the two papers are different: The present paper keeps the geometric style and is devoted mainly on maps from 4-manifolds to simply connected 4-manifolds, and however [3] carries the advantage of modern algebraic topology and is devoted to the computation of $D(M, N)$ for some classes of $(n-1)$-connected $2n$-manifolds, $n > 2$.

§2. Thom-Pontrjagin construction and presentation of $(n-1)$-connected $2n$-manifold.

All results in this section must be known. But we can not find the statements of Theorem 2.2 and Lemma 2.4 below in the literature, and therefore we provide proofs for them.

Let $f : S^{2n-1} \to S^n$ be any map with $y \in S^n$ a regular value. Then $W = f^{-1}(y) \subset S^{2n-1}$, an (n-1)-manifold, and $\tau = f^*(T_y S^n)$, the pull-back of the tangent space of S^n at y, together provide a framed normal bundle of W in S^{2n-1}. Call the pair (W, τ) a manifold in S^{2n-1} with a given framed normal bundle.

If both S^{2n-1} and S^n are oriented, then both W and the normal framed bundle are naturally oriented as follows: The normal space at each point of W has the orientation lifted from the orientation of the tangent space at $y \in S^n$, and W is oriented so that the orientations of W and its normal bundle give the orientation of S^{2n-1}.

On the other hand, for any framed normal bundle (W, τ), the ϵ-neighborhood W_ϵ of W in S^{2n-1} has a product structure $D^n \times W$ induced by the given framing. Let $p : D^n \times W \to D^n$ be the projection, $q : D^n \to D^n/\partial D^n = S^n$ be the quotient map, then we can extend $q \circ p : W_\epsilon \to S^n$ to a map $f : S^{2n-1} \to S^n$ by sending the remaining part of S^{2n-1} to the point $q(\partial D^n) \in S^n$.

Define

$$\mathcal{F}(n) = \{\ (n-1)\text{-manifold } W^{n-1} \subset S^{2n-1} \ \text{ with a framed normal bundle}\}$$

Note the underlying space of an element in $\mathcal{F}(n)$ does not need to be connected.

Let $\Omega(n)$ be the framed bordism classes of $\mathcal{F}(n)$. For each $s \in \mathcal{F}(n)$, we use $[s]$ for its framed bordism class.

Theorem 2.1. *The construction (Thom-Pontrjagin) above gives a 1-1 and onto map $T : \Omega(n) \to \pi_{2n-1}(S^n)$. (see [p.209 4] for a proof).*

For each $s = (W, \tau) \in \mathcal{F}(n)$, we will use $-s$ for $(-W, \tau)$. For $s' = (W', \tau') \in \mathcal{F}(n)$, which is disjoint from s, let $lk(s, s') = lk(W, W')$, and $s \sqcup s'$ be the disjoint union of s and s', which is again in $\mathcal{F}(n)$.

Theorem 2.2. *For any two disjoint $s_1, s_2 \in \mathcal{F}(n)$,*

$$T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2]) + lk(s_1, s_2)[s^n, s^n], \tag{2.1}$$

where s^n generates $\pi_n(S^n)$ and $[s^n, s^n] \in \pi_{2n-1}(S^n)$ is the Whitehead product.

A result similar to Theorem 2.2 is proved in [pp167-168 Lemma 2, 5], for diffeotopic classes (which are finer than framed bordism classes) of $n-$spheres (which form a proper subset of $n-$manifolds). Indeed Theorem 2.2 can be derived from [Lemma 2,

5] by first a projection and then some modifications. Since the statement of [Lemma 2, 5] involves many terminologies in homotopy theory and its proof also involves deep results of Smale and Kervaire, we prefer to give a direct proof of Theorem 2.2 as below.

Proof. We first verify that

$$T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2]) \qquad \text{if} \qquad lk(s_1, s_2) = 0. \tag{2.2}$$

If s_1 and s_2 are in the upper-hemisphere S^{2n-1}_+ and lower-hemisphere S^{2n-1}_- respectively, then clearly $T([s_1 \sqcup s_2]) = T([s_1]) + T([s_2])$. In general, by the condition $lk(s_1, s_2) = 0$ and standard arguments in geometric topology, one can construct a framed bordism $W_i \subset S^{2n-1} \times [0, 1]$ between $s_i \subset S^{2n-1} \times \{0\}$ and $s^*_i \subset S^{2n-1} \times \{1\}$, $i = 1, 2$, such that

(1) W_1 and W_2 are disjoint in $S^{2n-1} \times [0, 1]$,

(2) s^*_1 and s^*_2 are in $S^{2n-1}_+ \times \{1\}$ and $S^{2n-1}_- \times \{1\}$ respectively.

Then it follows that (2.2) still holds.

Now suppose $lk(s_1, s_2) = k \neq 0$.

First pick a $(2n-1)$-ball B^{2n-1} in S^{2n-1} which is disjoint from $s_1 \sqcup s_2$. Then pick two oriented n-balls $B_1, B_2 \subset \mathrm{int}B^{2n-1}$, with $lk(\partial B_1, \partial B_2) = k$.

Let $s'_i = (\partial B_i, \tau'_i) \in \mathcal{F}(n)$, where the normal framing τ'_i can be extended to B_i, $i = 1, 2$. Clearly we have

$$[s'_1] = [s'_2] = 0 \in \Omega(n), \tag{2.3}$$

and

$$lk(\pm s'_1, s'_2) = \pm k, \qquad lk(s_i, s'_j) = 0 \qquad i, j = 1, 2. \tag{2.4}$$

Then by (2.2), (2.3) and (2.4), we have

$$T([s_1 \sqcup s_2]) + T([-s'_1 \sqcup s'_2])$$

$$= T([(s_1 \sqcup s_2) \sqcup (-s'_1 \sqcup s'_2)]) = T([(s_1 \sqcup -s'_1) \sqcup (s_2 \sqcup s'_2)])$$

$$= T([s_1 \sqcup -s'_1]) + T([s_2 \sqcup s'_2]) = T([s_1]) + T([s_2]). \tag{2.5}$$

From the facts that $[s'_1] = [s'_2] = 0 \in \Omega(n)$, and their base spaces are the boundaries of n-balls, and the definitions of Thom-Pontrjagin construction and of Whitehead product [pp138-139, 6], one can verify that $T([s'_1 \sqcup s'_2]) = lk(s'_1, s'_2)[s^n, s^n]$. Hence

$T([-s_1' \sqcup s_2']) = -lk(s_1, s_2)[s^n, s^n]$. Substitute it into (2.5). We finish the proof of Theorem 2.2. □

Call M a pre-n-space, if M has the homotopy type of a space obtained from $V_{i=1}^m S_i^n$, the m-fold wedge of n-spheres with one point y in common, by attaching a $2n$-cell D^{2n} via an attaching map

$$g : S^{2n-1} = \partial D^{2n} \to V_{i=1}^m S_i^n. \tag{2.6}$$

So the homotopy type of M is determined by $g \in \pi_{2n-1}(V_{i=1}^m S_i^n)$. For short denote this space by $V_{i=1}^m S_i^n \cup_g D^{2n}$.

Suppose both S_i^n and D^{2n} are oriented. Let $y_i \in S_i^n$ be a point (other than the base point y). Make g transverse to y_i for each $i = 1, ..., m$ so that $K_i = g^{-1}(y_i)$ is an oriented $(n-1)$-manifold with a framed normal bundle in S^{2n-1}. The orientation of K_i is given by the orientations of $\partial D^{2n} = S^{2n-1}$ and S_i^n, and the orientation of the framed normal bundle of K_i is given by Thom-Pontrjagin construction, i.e., the pull-back of the orientation of the tangent space at y_i of S_i^n. The oriented framed (n-1)-manifolds $K = \{K_1, ..., K_m\}$ give a linking matrix $A = (a_{vw})$, where $a_{vw} = lk(K_v, K_w)$, the linking number of K_v and K_w. The manifold K_i with the given framed normal bundle gives an element $t_i \in \pi_{2n-1}(S^n)$.

So we have a set of invariants $(A; t_1, ..., t_m)$ of M. Denote

$$\mathcal{I}(m, n) = \{(A_{m\times m}; t_1, ..., t_m \in \pi_{2n-1}(S^n))\}, \tag{2.7}$$

where A is symmetric if n is even and is anti-symmetric if n is odd. Moreover M here has the homotopy type of a manifold, thus A is unimodular.

Theorem 2.3. *There is a 1-1 correspondence $H : \pi_{2n-1}(V_{i=1}^m S_i^n) \to \mathcal{I}(m, n)$. Hence*

(1) $(A; t_1, ..., t_m)$ is a complete homotopy type invariant of the pre-n-space M.

(2) Moreover if $n = 2$, then A itself is a complete homotopy type invariant.

Proof. The result is known. For (1) see [p. 128, 5]. For (2) see [p. 23, 7], and (2) is also derived from (1) by the fact that $\pi_3(S^2) = Z$ is torsion free and hence a_{ii} determines t_i. The proof is an application of Thom-Pontrjagin construction. See [p. 23, 7]. □

Lemma 2.4. *The linking matrix A provided by presentation $V_{i=1}^m S_i^n \cup_g D^{2n}$ is the intersection matrix under the algebraic dual basis α^* of $\alpha = ([S_1^n], ..., [S_m^n])$.*

Proof. Now we can further homotope g above so that $K_i = g^{-1}(y_i) \subset S^{2n-1}$ is connected. Let $F_i' \subset D^{2n}$ be an oriented n-chain bounded by K_i. Let F_i be the cycle in $M = V_{i=1}^m S_i^n \cup_g D^{2n}$, defined by F_i'/K_i, where $g(K_i) = y_i$. Then

(a) $S_i^n \cap F_j$ is a point if $i = j$ and is empty if $i \neq j$,

(b) the intersection matrix of M defined by cap product under the basis $\alpha_* = (F_1, ..., F_m) \subset \bar{H}_n(M)$ is the linking matrix A provided by g.

Let α^* be the Poincaré dual basis of α_*. (b) implies that the intersection matrix of M under the basis α^* is A and then (a) implies that α^* is the algebraic dual of α. $\square$

§3. Proof of Theorem 1.

Recall that M and L are closed $(n-1)$-connected $2n$-manifolds with presentations

$$M = V_{i=1}^{m} S_i^n \cup_g D^{2n} \qquad \text{and} \qquad L = V_{j=1}^{l} S_j^n \cup_h D^{2n}. \tag{3.1}$$

We can consider

$$\alpha = \{S_i^n, i = 1, ..., m\}, \qquad \text{and} \qquad \beta = \{S_j^n, j = 1, ..., l\} \tag{3.2}$$

as bases for either $H_n(M)$ and $H_n(L)$, or $H_n(V_{i=1}^{m} S_i^n)$ and $H_n(V_{j=1}^{l} S_j^n)$.

Proposition 3.1. *Let $f : V_{i=1}^{m} S_i^n \to V_{j=1}^{l} S_j^n$ be a map such that $f_*\alpha = \beta P^{\perp}$. Then for the element $g \in \pi_{2n-1}(V_{i=1}^{m} S_i^n)$ corresponding to $(A; t_1, ..., t_m) \in \mathcal{I}(m, n)$, the element $f_*(g) \in \pi_{2n-1}(V_{j=1}^{l} S_j^n)$ is corresponding to $(A'; t'_1, ..., t'_l) \in \mathcal{I}(l, n)$, where*

$$A' = P^{\perp} A P \tag{3.3}$$

and

$$t'_r = \sum_v p_{vr} t_v + (\sum_v \frac{1}{2} p_{vr}(p_{vr} - 1) a_{vv} + \sum_{v<w} p_{vr} p_{wr} a_{vw})[s^n, s^n]. \tag{3.4}$$

Proof. Call a map from an n-disc D^n to an n-sphere S^n with base point z a boundary pinch if the interior of D^n is mapped onto $S^n - z$ homeomorphically and ∂D^n is mapped onto z. If both D^n and S^n are oriented, we can say a boundary pinch is of degree 1 or degree -1.

Recall $P = (p_{ij})_{m\times l}$, then $P^{\perp} = (p_{ji})_{l\times m}$.

For each i, take $\sum_j |p_{ij}|$ disjoint n-discs in S_i^n away from the common wedge point $y \in V_{i=1}^{m} S_i^n$. Each n-disc has the induced orientation of S_i^n. Then map $|p_{ij}|$ discs to S_j^n via boundary pinches of degree 1 if p_{ij} is positive or -1 otherwise. Doing this for all i, and finally mapping the complement of all these $\sum_{i,j} |p_{ij}|$ discs to the base point $z \in V_{j=1}^{l} S_j^n$, we get a map $f' : V_{i=1}^{m} S_i^n \to V_{j=1}^{l} S_j^n$ and clearly $f'_*\alpha = \beta P^{\perp}$ on $H_n = \pi_n$. Since the homotopy group π_i vanishes for $0 < i < n$, up to homotopy, we may assume that $f = f'$.

Let $z_j \in S_j^n - z$. Note $f^{-1}(z_j)$ consists of $\sum_i |p_{ij}|$ points, and $|p_{ij}|$ points of them lie in S_i^n with sign 1 if p_{ij} is positive or sign -1 otherwise. Suppose g is transverse to

$y_i \in S_i^n - y$. Then the oriented $(n-1)$-manifold $K_i = g^{-1}(y_i)$ and its lifted framed normal bundle $g^*(T_{y_i}S_i^n)$ provide an element $s_i \in \mathcal{F}(n)$. Moreover $A = (lk(K_i, K_j))$ and $t_i = T([s_i])$ provide the set of invariants $(A; t_1, ..., t_m)$. Now suppose $f \circ g$ is transverse to z_r and

$$(f \circ g)^{-1}(z_r) = g^{-1}(f^{-1}(z_r)) = \sqcup_{v=1}^m \sqcup_{e=1}^{|p_{vr}|} \mathrm{sign}(p_{vr}) K_v^e, \tag{3.5}$$

where K_v^e is the preimage of a point in $f^{-1}(z_r) \cap S_v^n$ under g. Let $s_v^e \in \mathcal{F}(n)$ be K_v^e with its lifted framed normal bundle. We have

$$t'_r = T([\sqcup_{v=1}^m \sqcup_{e=1}^{|p_{vr}|} \mathrm{sign}(p_{vr}) s_v^e]). \tag{3.6}$$

Then

$$lk(s_v^e, s_w^{e'}) = lk(K_v^e, K_w^{e'}) = lk(K_v, K_w) = a_{vw} \text{ and } T([s_v^e]) = T([s_v]) = t_v. \tag{3.7}$$

Now we use abbreviations as below

$$p_{vr}K_v = \sqcup_{e=1}^{|p_{vr}|} \mathrm{sign}(p_{vr}) K_v^e, \qquad p_{vr}s_v = \sqcup_{e=1}^{|p_{vr}|} \mathrm{sign}(p_{vr}) s_v^e. \tag{3.8}$$

Now (3.3) follows from (3.5), (3.7), (3.8) and definition, since

$$A' = (a'_{rs}) = (lk((f \circ g)^{-1}(z_r), (f \circ g)^{-1}(z_s))) = (lk(\sqcup_v p_{vr}K_v, \sqcup_w p_{ws}K_w))$$

$$= (\sum_{v,w} p_{vr}p_{ws} lk(K_v, K_w)) = (\sum_{v,w} p_{vr}p_{ws}a_{vw}) = P^\perp AP. \tag{3.9}$$

By (3.6), (3.7), (3.8), and by applying Theorem 2.2 to $T([\sqcup_v p_{vr}s_v])$ and $T([p_{vr}s_v])$, (3.4) follows, since

$$T([\sqcup_v p_{vr}s_v]) = \sum_v T([p_{vr}s_v]) + \sum_{v<w} p_{vr}p_{wr} lk(s_v, s_w)[s^n, s^n]$$

$$= \sum_v p_{vr}T([s_v]) + (\sum_v \frac{1}{2} p_{vr}(p_{vr} - 1)a_{vv} + \sum_{v<w} p_{vr}p_{wr}a_{vw})[s^n, s^n]. \tag{3.11}$$

Proposition 3.1 is proved. $\square$

Proof of Theorem 1.

(1) The sufficient part. We are going to show that if the complete homotopy invariants of M and L, and the matrix P satisfy equations (1.4) and (1.5), then there is a map $f : M \to L$ of degree k such that $f_*\alpha = \beta P^{\perp}$.

Let $J = (J_1, ..., J_l) = h^{-1}(z_1, ..., z_l) \subset B^{2n-1} \subset S^{2n-1}$ be the (n-1)-manifold provided by $h : S^{2n-1} \to V_{j=1}^{l} S_j^n$ and B^{2n-1} is a (2n-1)-ball. Let $p_k : S^{2n-1} \to S^{2n-1}$ be a branched covering of degree k over a standard sphere $S^{2n-3} \subset S^{2n-1}$ which misses B^{2n-1}. Then $p_k^{-1}(J) = (J^1, ..., J^k)$, each $J^i = (J_1^i, ..., J_l^i)$ lies in a homeomorphic lift B_i^{2n-1} of B^{2n-1}. It follows that

$$lk(J_v^i, J_w^i) = lk(J_v, J_w) \text{ and } lk(J_v^i, J_w^j) = 0 \text{ if } i \neq j. \tag{3.12}$$

Now consider the composition

$$S^{2n-1} \xrightarrow{p_k} S^{2n-1} \xrightarrow{h} V_j S_j^n. \tag{3.13}$$

By (3.12), the discussion above and Theorem 2.2, it is easy to verify that the element in $\mathcal{I}(l, n)$ corresponding to $[h \circ p_k] = k[h] \in \pi_{2n-1}(V_{j=1}^{l} S_j^n)$ is $(kB; ku_1, ..., ku_l)$.

Let $f : V_{i=1}^{m} S_i^n \to V_{j=1}^{l} S_j^n$ be a map such that $f_*(\alpha) = \beta P^{\perp}$. Then by (3.3) and (3.4) of Proposition 3.1, we have

$$[f \circ g] = [h \circ p_k], \tag{3.14}$$

that is

$$f_*[g] = k[h] \in \pi_{2n-1}(V_{j=1}^{l} S_j^n). \tag{3.15}$$

By (3.15), $f : V_i S_i^n \to V_j S_j^n$ can be extended to a map, still denoted by f,

$$M = V_{i=1}^{m} S_i^n \cup_g D^{2n} \to N = V_{j=1}^{l} S_j^n \cup_h D^{2n}. \tag{3.16}$$

Clearly $f_*(\alpha) = \beta P^{\perp}$. It is also easy to verify that f is of degree k, since p_k is of degree k. $\square$

(2) The necessary part. Suppose there is a map $f : M \to L$ of degree k such that $f_*\alpha = \beta P^{\perp}$. We are going to show that the complete homotopy invariants of M and L and the matrix P satisfy equations (1.4) and (1.5).

Up to homotopy, we may assume that $f(V_i S_i^n) \subset V_j S_j^n$ and the restriction on $V_i S_i^n$ is the same as the map f' given in the proof of Proposition 3.1. Then by Proposition 3.1, the invariant $(A'; t_1', ..., t_l')$ associated with the map $f \circ g$ is given by (3.3) and (3.4).

Since the kernel of

$$i_* : \pi_{2n-1}(V_j S_j^n) \to \pi_{2n-1}(V_j S_j^n \cup_h D^{2n}) = \pi_{2n-1}(V_j S_j^n)/[h] \tag{3.18}$$

induced by the inclusion $i : V_j S_j^n \to V_j S_j^n \cup_h D^{2n}$ is the cyclic group $< [h] >$, and $[i \circ f \circ g] = 0$, we have $[f \circ g] = d[h]$ for some integer d. It follows that (1.4) and (1.5) hold if we replace k by d.

Under the algebraic dual bases α^* and β^* of α and β, the intersection matrices are A and B for M and L respectively by Lemma 2.4. Moreover $f^*\beta^* = \alpha^* P$. Since f is of degree k, we have $P^\perp AP = kB$ by Theorem 2 (note the proofs of Theorem 2 and Lemma 2.4 are independent of Theorem 1). So we have $d = k$. $\square$

(3) The 1-1 correspondence. Let $\mathcal{P} = \{f : M \to L | f_*\alpha = \beta P^\perp\}$. Since M and L are $(n-1)$-connected $2n$-manifolds, for any $f \in \mathcal{P}$, we may assume that up to homotopy

$$f', f : M = V_{i=1}^m S_i^n \cup_g D^{2n} \to N = V_{j=1}^l S_j^n \cup_h D^{2n},$$

differ only in the interior of D^{2n}, where f' is constructed in the proof of Proposition 3.1. Now the restrictions of f and f' on D^{2n} give a map $C(f, f') : S^{2n} \to L$, which presents an element in $\pi_{2n}(L)$. Then the verification that $f \mapsto C(f, f')$ for $f \in \mathcal{P}$ induces a 1-1 correspondence $\{[f] | f \in \mathcal{P}\} \to \pi_{2n}(L)$ is routine.

(The finiteness of $\pi_{2n}(L)$ follows from [p. 45, Kahn, Math. Ann., Vol. 180]). $\square$

§4. Proof of Theorem 2.

Recall we have

$$\alpha^{*\perp} \cup \alpha^* = A[M] = (a_{ij})[M], \qquad \beta^{*\perp} \cup \beta^* = B[L] = (b_{ij})[L]. \tag{4.0}$$

Suppose $f : M \to L$ is of degree k such that $f^*(\beta^*) = \alpha^* P$. Then we have

$$f^*([L]) = k[M]. \tag{4.1}$$

By (4.0) and (4.1), we have

$$kB[M] = (b_{ij}k[M]) = (f^*(b_{ij}[L])) = (f^*(b_i^* \cup b_j^*))$$

$$= (f^*(b_i^*) \cup f^*(b_j^*)) = (f^*(\beta^*))^\perp \cup (f^*\beta^*) = (\alpha^* P)^\perp \cup \alpha^* P \tag{4.2}$$

$$= P^\perp(\alpha^{*\perp} \cup \alpha^*)P = P^\perp AP[M].$$

That is

$$P^{\perp}AP = kB. \tag{4.3}$$

For any $b_i^*, b_j^* \in \beta^* \subset \bar{H}^n(L)$, we have

$$X_M(f^*(b_i^*), f^*(b_j^*))[M] = f^*(b_i^*) \cup f^*(b_j^*)$$

$$= f^*(b_i^* \cup b_j^*) = f^*(b_{ij}[L]) = b_{ij} f^*([L]) = b_{ij}[M]. \tag{4.4}$$

Hence the restriction of X_M on the subgroup $f^*(\bar{H}^n(L)) \subset \bar{H}^n(M)$ is isomorphic to X_L. In particular it is still unimodular. By orthogonal decomposition lemma [p.5, 8],

$$X_M = X_{f^*\bar{H}^n(L)} \oplus X_{H'} = X_L \oplus X_{H'},$$

where H' is the orthogonal complement of $f^*\bar{H}^n(L)$ and $X_{H'}$ is the restriction of X_M on H'. $\square$

§5. Proof of Theorem 3.

Proposition 5.1. *Let M and L be two closed oriented 4-manifolds satisfying*

$$X_M = X_L \oplus C. \tag{5.1}$$

If L is simply connected, then there is a map $g : M \to L$ of degree 1.

Proof. In the whole proof, the intersection forms are understood in homology rings defined by cap product. We also use the same symbol for a surface and its homology class. The proof is divided into three steps:

(1) First prove Proposition 5.1 in the case M is also simply connected.

Suppose $X_M = X_L \oplus C$. Clearly C is symmetric and unimodular, therefore C represents the intersection form of a simply connected 4-manifold Q by Freedman's work [9]. So we have $X_M = X_L \oplus X_Q = X_{L\#Q}$. By Whitehead's theorem [II. Theorem 2.1, 7], M and $L\#Q$ are homotopy equivalent. Then there is a homotopy equivalence $h : M \to L\#Q$, which is of degree one. There is also an obvious degree one pinch $p : L\#Q \to L$. Then $p \circ h : M \to L$ is a map of degree one.

(2) Then prove Proposition 5.1 in the case $X_M = X_L$.

Let $\mathcal{F} = \{F_j,\ j = 1, ..., m\}$ be a set of oriented embedded surfaces which provide a basis of $\bar{H}_2(M)$.

Suppose M is not simply connected (otherwise it is proved in (1)). Let $\{C_i,\ i = 1, ..., l\}$ be a set of disjoint simple closed curves which generate $\pi_1(M)$ satisfying

$$(\cup_i N(C_i)) \cap (\cup_j F_j) = \emptyset, \tag{5.2}$$

where $N(C_i) = C_i \times D^3$ is a regular neighborhood of C_i in M. Let M^* be the simply connected 4-manifold obtained from M by surgery on $\{C_i\}$. Precisely

$$M^* = (M - \text{int} \cup_i C_i \times D^3) \cup_{\{h_i\}} (\cup_i D_i^2 \times S^2), \tag{5.3}$$

where the gluing map

$$h_i : S_i^1 \times S^2 = \partial D_i^2 \times S^2 \to \partial(C_i \times D^3) = C_i \times S^2 \tag{5.4}$$

is an orientation reversing homeomorphism.

Since $\mathcal{F}$ is linear independent in $\bar{H}_2(M)$, it is easy to see that $\mathcal{F}$ is still linear independent in $H_2(M^*)$. Hence $\mathcal{F}$ is a basis of a submodule H of $H_2(M^*)$ and the restriction of the intersection form X_{M^*} on H is still X_M, which is isomorphic to X_L. In particular the restriction of X_{M^*} on H is still unimodular. By [p.5, 8], we have

$$X_{M^*} = X_M \oplus X_{H'} = X_L \oplus X_{H'}, \tag{5.5}$$

where H' is the orthogonal complement of H in $H_2(M^*)$, and

$$H_2(M^*) = H \oplus H' = H_2(L) \oplus H'. \tag{5.6}$$

By part (1), we have a simply connected 4-manifold Q with $H_2(Q) = H'$ and a degree one map

$$M^* \xrightarrow{h} L\#Q \xrightarrow{p} L, \tag{5.7}$$

where h is a homotopy equivalence and p pinches Q. Let $f = p \circ h$. For

$$f_* : H_2(M^*) = \pi_2(M^*) \to H_2(L) = \pi_2(L), \tag{5.8}$$

we have $H' = \text{Ker} f_*$.

Now we will transfer the degree one map in (5.7) to a degree one map $M \to L$.

Since $*_i \times S^2$ is disjoint from all F_j, $*_i \times S^2 \in H'$ for each i, where $*_i \in D_i^2$. Then by (5.8) $f(*_i \times S^2)$ is null homologous, and therefore is null homotopic. Now we can homotope f so that $f(D_i^2 \times S^2) = z$ for each i by first shrinking $D_i^2 \times S^2$ to its core $*_i \times S^2$, then shrinking $*_i \times S^2$ to z.

Note $M^* - \text{int} \cup_i D_i^2 \times S^2 = M - \text{int} \cup_i C_i \times D^3$. Since $f : M^* \to L$ is of degree one, $f(D_i^2 \times S^2) = z$ for all i, we have that the restriction

$$f| : (M - \text{int} \cup_i C_i \times D^3, \cup_i C_i \times S^2) \to (L, z) \tag{5.9}$$

is a map between pairs of degree 1.

Now we can extend $f|$ to $g : M \to L$ by sending each $C_i \times D^3$ to z. Since g sends $C_i \times D^3$ to z, the extension $g : M \to L$ is of degree 1.

(3) Prove Proposition 5.1 from (1) and (2).

By [9], there is a simply connected 4-manifold Q with $X_Q = X_M = X_L \oplus C$. Then there is a degree one map from M to Q by (2) and a degree one map from Q to L by (1). Hence there is a degree one map form M to L. $\square$

Proof of Theorem 3. The necessary part follows from Theorem 2, and the proof of the sufficient part is divided into two steps.

(1) Suppose first that M is also simply connected. We may suppose that M and L have presentations (3.1), and the bases α and β for $\bar{H}_2(M)$ and $\bar{H}_2(L)$ are chosen as in (3.2). We may further suppose that α^* and β^* in Theorem 3 are the algebraic dual bases of α and β respectively.

Then $f^*(\beta^*) = \alpha^* P$ implies that $f_*(\alpha) = \beta P^\perp$, and by Lemma 2.4, the linking matrices for M and L in the presentations (3.1) are A and B in Theorem 3 respectively. Hence (1.4) in Theorem 1 holds. Since $n = 2$, by Theorem 2.3 (2), (1.5) is covered by (1.4). Hence there is a map $f : M \to L$ of degree k realizing P by Theorem 1.

(2) If M is not simply connected, let Q be a simply connected 4-manifold with $X_Q = X_M$. By Proposition 5.1, there is a degree one map $h : M \to Q$ which induces an isomorphism $h_* : \bar{H}_2(M) \to H_2(Q)$, hence also induces an isomorphism $h^* : H^2(Q) \to \bar{H}^2(M)$. Under the basis ${h^*}^{-1}(\alpha^*)$, the intersection matrix of X_Q is A. Now ${h^*}^{-1}(\alpha^* P) = {h^*}^{-1}(\alpha^*)P$ and $P^\perp AP = kB$.

By step (1) there is a map $f' : Q \to L$ of degree k such that ${f'}^*\beta^* = {h^*}^{-1}(\alpha^* P)$. And finally let $f = f' \circ h$. f is of degree k and $f^*\beta^* = \alpha^* P$. $\square$

§6. Proof of the corollaries.

Proof of Corollary 1. The first claim in Corollary 1 follows from Theorem 2 and the arguments in the proof of Proposition 5.1 (1).

For the remaining, since $\beta_0(M) = \beta_0(L) = \beta_4(M) = \beta_4(L) = 1$ and $\beta_1(M) = \beta_1(L) = \beta_3(M) = \beta_3(L) = 0$, $\chi(M) = \chi(L)$ implies that $\beta_2(M) = \beta_2(L)$, where β_i is the i-th Betti number. Since f is of degree one, by Theorem 3, the intersection forms of M and L must be isomorphic. By [II.Theorem 2.1, 7], M and L are homotopy equivalent. (If the intersection form X_M is even, M and L are homeomorphic by [9]). $\square$

Proof of Corollary 2. It is known that $\pi_{2n-1}(S^n) =< \nu > \oplus G$, where ν has infinite order if n is even and is 0 if n is odd, and G is a finite group (see [p.318 and p.325, 6]). Moreover if n is even and $t \in \pi_{2n-1}(S^n)$, then $t = \lambda H(t)\nu + \mu$, where $\mu \in G$, the Hopf invariant $H(t)$ is the self-linking number defined by the map $t : S^{2n-1} \to S^n$

and $\lambda = 1$ if $n = 2, 4, 8$ and $\lambda = 1/2$ otherwise (see [p.326, 6]). It is also known that $H([s^n, s^n]) = 2$ ([p.336, 6]).

Let $(A; t_1, ..., t_m)$ be the complete homotopy invariant of M.

Now $t_v = \lambda H(t_v)\nu + \mu_v = a_{vv}\lambda\nu + \mu_v$ and $[s^n, s^n] = 2\lambda\nu + \mu$, where $\mu_v, \mu \in G$. Let $T(n)$ be the order of the torsion part of $\pi_{2n-1}(S^n)$ and k be a multiple of $2T(n)$ if $T(n)$ is even, or a multiple of $T(n)$ otherwise. Let $P = kI_m$, where I_m is the m by m unit matrix. Now we substitute all these information into each side of (1.4) and (1.5) in Theorem 1. We first have

$$P^{\perp}AP = k^2 A. \tag{6.1}$$

Now the right side of (1.5) in Theorem 1 can be reduced to

$$\sum_v p_{vr}t_v + (\sum_v \frac{1}{2}p_{vr}(p_{vr}-1)a_{vv} + \sum_{v<w} p_{vr}p_{wr}a_{vw})[s^n, s^n]$$

$$= kt_r + \frac{1}{2}k(k-1)a_{rr}[s^n, s^n]$$

$$= k(a_{rr}\lambda\nu + \mu_r) + \frac{1}{2}k(k-1)a_{rr}(2\lambda\nu + \mu)$$

$$= ka_{rr}\lambda\nu + k(k-1)a_{rr}\lambda\nu = k^2 a_{rr}\lambda\nu \tag{6.2}$$

On the other hand, clearly

$$k^2 t_r = k^2(a_{rr}\lambda\nu + \mu_r) = k^2 a_{rr}\lambda\nu. \tag{6.3}$$

(6.1) implies that (1.4) of Theorem 1 holds for k^2, and (6.2) and (6.3) imply that (1.5) of Theorem 1 holds for k^2. By Theorem 1 there is a map of degree k^2 from M to itself realizing P.

In the case $n = 2$, we have $T(2) = 1$. We have finished the proof. $\square$

Proof of Corollary 3. Let P be the matrix realized by f. By (1.6) in Theorem 2 we have

$$P^{\perp}AP = kA, \tag{6.4}$$

where both A, P are square matrices of order $\beta_n(M)$. By taking the determinants we have $|P|^2|A| = k^{\beta_n(M)}|A|$. Thus $|P|^2 = k^{\beta_n(M)}$. Since $\beta_n(M)$ is odd, k itself must be a perfect square. $\square$

Proof of Corollary 4. From the equation $P^\perp AP = kB$, i.e., (1.6) in Theorem 2, we have

$$kb_{ii} = \sum_{r,s} a_{rs}p_{ri}p_{si} = \sum_r a_{rr}p_{ri}^2 + \sum_{r<s}(a_{rs}p_{ri}p_{si} + a_{sr}p_{si}p_{ri}). \tag{6.5}$$

Since X_M is even, each a_{rr} is even. Since $a_{rs} = \pm a_{sr}$, the right side of the equation is even. Since X_L is not even, b_{ii} is odd for some i. It follows k is even. □

Proof of Corollary 5. If M dominates L, then the rank of the intersection form of L is at most the rank of the intersection form of M by Theorem 2. For a given positive integer m, there are only finitely many non-isomorphic forms of rank m [p.18 and p.25, 8]. Since the torsion part of $\pi_{2n-1}(S^n)$ is finite, the homotopy types of possible L are finite. In the case of 4-manifolds, there are at most two simply connected 4-manifolds with a given form X [9]. So the conclusion follows. □

§7. Examples.

Now we will apply the results above to some simple examples of 4-manifolds.

Let F_g, CP^2 and T^4 be the surface of genus g, complex projective plane and 4-dimensional torus respectively.

Let

$$I_l = \oplus_l(1), \qquad A_l = \oplus_l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P_{l,k} = \oplus_l \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example 1. Any closed orientable 4-manifold M with non-trivial intersection form dominates CP^2.

Verification: Suppose M has intersection matrix $A = (a_{ij})_{m\times m}$. If $a_{ii} \neq 0$ for some i, then let $c_i = 1$ and $c_r = 0$ for $r \neq i$. If $a_{ii} = 0$ for all i, then pick any $a_{ij} \neq 0$, let $c_i = c_j = 1$ and $c_r = 0$ for $r \neq i, j$. Let $C = (c_i)_{m\times 1}$. Then

$$C^\perp AC = k(1).$$

Then $k = 2a_{ij}$ if $i \neq j$ and $k = a_{ii}$ otherwise. By Theorem 3, there is a map $f : M \to CP^2$ of degree $k \neq 0$.

Example 2. Suppose M is a simply connected 4-manifold. (1) If $P_{m\times 1}$ is realized by a map $f : M \to CP^2$, then any map $M \to CP^2$ realizing P is homotopic to f. (2) If $P_{m\times 2}$ is realized by a map $f : M \to S^2 \times S^2$, then all maps $M \to S^2 \times S^2$ realizing P have 4 homotopy classes. (3) There are infinitely many homotopy classes of maps $(S^2 \times S^2)\#(S^3 \times S^1) \to S^2 \times S^2$ realizing I_2.

Verification: By the exact sequence of fibration $S^5 \to CP^2$, we have $\pi_4(CP^2) = \pi_4(S^5) = 0$. Moreover $\pi_4(S^2 \times S^2) = \pi_4(S^2) \oplus \pi_4(S^2) = Z_2 \oplus Z_2$. Then (1) and (2) follow from Theorem 1. For (3), consider the composition

$$(S^2\times S^2)\#(S^3\times S^1)\xrightarrow{p_1}(S^2\times S^2)V(S^3\times S^1)\xrightarrow{p_2}(S^2\times S^2)VS^3\xrightarrow{g_k}S^2\times S^2,$$

where p_1 pinches the 3-sphere separating $S^2\times S^2$ and $S^3\times S^1$ to a point to get the one point union of $S^2\times S^2$ and $S^3\times S^1$, $p_2|S^3\times S^1 : S^3\times S^1\to S^3$ is the projection and $p_2|S^2\times S^2$ is the identity, $g_k|S^3$ sends S^3 to the first factor of $S^2\times S^2$ with Hopf invariant k and $g_k|S^2\times S^2 : S^2\times S^2\to S^2\times S^2$ realizes I_2. Let $f_k = g_k\circ p_2\circ p_1$, then $\{f_k\}$ forms infinitely many homotopy classes realizing I_2.

Example 3. $D(CP^2\#(-CP^2), S^2\times S^2) = D(S^2\times S^2, CP^2\#(-CP^2)) = 2Z$ and $D(CP^2\#CP^2, S^2\times S^2) = D(S^2\times S^2, CP^2\#CP^2) = D(CP^2\#(-CP^2), CP^2\#CP^2)$ $= D(CP^2\#CP^2, CP^2\#(-CP^2)) = \{0\}$.

Verification: Let $P=\begin{pmatrix} a & c\\ b & d\end{pmatrix}$. Since

$$P^\perp BP=\begin{pmatrix} a^2-b^2 & ac-bd\\ ac-bd & c^2-d^2\end{pmatrix},$$

one can verify easily that there is no solution P for $P^\perp BP = kI_2$ for any non-zero integer k and no solution P for $P^\perp BP = kA_1$ for any odd integer k. Moreover $P=\begin{pmatrix} 1 & k\\ 1 & -k\end{pmatrix}$ is a solution for $P^\perp BP = 2kA_1$.

Since I_2, B and A_1 are the intersection forms of $CP^2\#CP^2$, $CP^2\#(-CP^2)$ and $S^2\times S^2$ respectively, $D(CP^2\#(-CP^2), CP^2\#CP^2) = \{0\}$ and $D(CP^2\#(-CP^2), S^2$ $\times S^2) = 2Z$ follow from Theorem 3. The verification for the remaining is similar.

Example 4. $D(T^4, \#_3 S^2\times S^2) = Z$. That is for any k, there is a map of degree k from 4-torus to the connected sum of 3 copies of $S^2\times S^2$. In general $D(F_s\times F_r, \#_q S^2\times S^2) = Z$ if $q\le 2rs+1$.

Verification: Note $T^4 = S^1_1\times S^1_2\times S^1_3\times S^1_4$. Let $a_i\in H^1(T^4)$ be the algebraic dual of S^1_i. Let $c_{ij} = a_i\cup a_j$. Then under the basis $(c_{12}, c_{34}, c_{13}, c_{24}, c_{14}, c_{23})$ of $H^2(T^4)$, the intersection matrix is A_3, which is the same as the matrix of $\#_3 S^2\times S^2$ under the obvious basis. Then it is easy to verify

$$P^\perp_{3,k} A_3 P_{3,k} = kA_3.$$

Hence there is a map $f_k : T^4\to \#_3 S^2\times S^2$ of degree k by Theorem 3.

The verification of the second part is similar.

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